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Analytical solution of the $\mu(I)$ -rheology for fully developed granular flows in simple configurations

Merline Tankeo · Patrick Richard and
Édouard Canot

Abstract Using the $\mu(I)$ continuum model recently proposed for dense granular flows, we study theoretically steady and fully developed granular flows in two configurations: a plane shear cell and a channel made of two parallel plates (Poiseuille configuration). In such a description, the granular medium behaves like a fluid whose viscosity is a function of the inertia. In the shear plane geometry our calculation predicts that the height of the shear bands scales with $U_0^{1/4} P_0^{1/2}$, where U_0 is the velocity of the moving plate and P_0 the pressure applied at its top. In the Poiseuille configuration, the medium is sheared between the lateral boundaries and a plug flow is located in the center of the channel. The size of the plug flow is found to increase for a decreasing pressure gradient. We show that, for small pressure gradient, the granular material behaves like a Bingham plastic fluid.

1 Introduction

Granular flows [1] are of important scientific interest because of their complex nature as well as their wide occurrence in industry and in environment. Unlike classical fluid flows, they display different behaviors in different flow regimes thus making difficult a complete and general constitutive law from being derived.

Three flow regimes are generally reported in the literature. In case of compact, slowly sheared flows, the grains experience enduring contacts. They dissipate energy by internal friction, so the constitutive law is plastic-like. In case of dilute,

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rapidly sheared and agitated flows, the granular materials interact mainly through collisions. The constitutive law can be deduced from the kinetic theory of a gas of inelastic grains [2]. In the intermediate flow regime, for example dense granular flows down an inclined plane, the granular materials are dense as well as rapid, and subject to both frictional and collisional stresses.

In the last decade, significant theoretical progresses [3,4] has been made for the latter regime. Those approaches consist in describing the granular medium as an incompressible fluid whose behavior is captured by a purely local rheology (called the $\mu(I)$ rheology) that can be used to write the stresses in balance equations:

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P + \frac{1}{\rho} \nabla \cdot \boldsymbol{\tau}, \\ \nabla \cdot \mathbf{u} = 0. \end{cases} \quad (1)$$

In those equations, P is the pressure, $\boldsymbol{\tau}$ the deviatoric stress tensor, \mathbf{u} the velocity and ρ the bulk density. Such a rheology is able to reproduce observations from a great variety of experimental and numerical setups [3–9]. It is based on a coulombic friction model, and relates the value of the effective coefficient of friction μ (i.e. the ratio of tangential to normal stresses) to the non-dimensional inertial number I that compares the typical time scale of microscopic rearrangements with the typical time scale of macroscopic deformations:

$$\frac{|\boldsymbol{\tau}|}{P} = \mu(I) \quad \text{with} \quad I = \frac{|\dot{\gamma}| d}{\sqrt{P/\rho_s}}, \quad (2)$$

where $|\boldsymbol{\tau}| = \sqrt{\frac{1}{2} \tau_{ij} \tau_{ij}}$ is the deviatoric stress tensor norm, d is the particle diameter, ρ_s is the particle density and $\dot{\gamma}$ the shear rate. Note that the inertial number I is the square root of the Savage number [10] also called the Coulomb number [11]. It has been empirically shown [12] that, for dense granular flows, the effective coefficient of friction μ of the system can be expressed by the following expression:

$$\mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_0/I + 1}. \quad (3)$$

In the previous expression, μ_s is the threshold value for the quasi-static regime ($I \rightarrow 0$). It corresponds to the angle of repose of the material. Therefore, the material flows only if the yield criterion $|\boldsymbol{\tau}| > \mu_s P$ is satisfied. Below this threshold, the system behaves locally as a rigid body. In strongly sheared regimes ($I \gg 1$), $\mu(I)$ grows asymptotically towards μ_2 . In Eq. (3), the values of the coefficients are material-dependent, for example the values for the spherical glass bead used in [12] are $\mu_s = \tan(20.9^\circ)$, $\mu_2 = \tan(32.76^\circ)$ and $I_0 = 0.279$. If the inertial number I is much lower than I_0 ($I \ll I_0$) the coefficient of friction can be approximated by the following (simpler) expression:

$$\mu(I) \approx \mu_s + (\mu_2 - \mu_s) \frac{I}{I_0}, \quad (4)$$

with $\mu_2 > \mu_s$. Recently it has been shown [13,14] that the tensorial extension [6] of the $\mu(I)$ rheology is questionable since stress and strain tensors are not always aligned. Therefore the $\mu(I)$ should be applied only to monodirectional flows. In such a case Eq. 1 becomes

$$\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \frac{1}{\rho} \frac{\partial \tau}{\partial y}, \quad (5)$$

where τ is the shear stress.

Another quantity of interest is the packing fraction Φ , which has been found to decrease when the inertial number I increases [15].

$$\Phi = \Phi_{max} - \zeta I, \quad (6)$$

where Φ_{max} is the maximum packing fraction of the system and ζ is a positive constant typically equal to 0.2. The latter equation is only valid for small values of I since it leads to negative packing fractions for $I > \Phi_{max}/\zeta$. This is consistent with the restriction of the $\mu(I)$ rheology to dense flows where relatively small values of I are expected. However, for sake of simplicity, in the following, we will not take into account the latter equation and assume that the packing fraction does not depend on I : $\Phi = \Phi_{max}$. This assumption will be discussed in the last section of the paper. For systems made of monodispersed spherical glass beads, $\Phi_{max} \approx 0.6$.

As shown above, the $\mu(I)$ -rheology is based on a phenomenological approach. Other models, based on different theoretical backgrounds [16–21] can be found in the literature, but it has the advantage to be simple and to compare well against many experiments. It should be however pointed out that this rheology is purely local, i.e. the shear stress depends only on the local shear rate and pressure. Hence, it does not include long range correlations, which are prevalent near the jamming point [22–24]. A possible way to overcome this flaw, consists in introducing non-local effects in such models (see e.g. [25, 26]). Another questionable point is that such a rheology does not use the notion of granular temperature which is at the base of the kinetic theory [2] even in the case of dense flows. Some discrepancies with experiments and simulations are also found in the case of dilute granular flows or important inclination angles [13, 14, 27, 28]. Moreover the influence of the fluctuating energy flux is not taken into account. This point is problematic especially close to boundaries. In spite of its flaws, the $\mu(I)$ rheology emerges so far as a reliable description of granular flows, at least if they are dense.

In this article, we use the $\mu(I)$ rheology to solve analytically the conservation of momentum equation in the case of two-dimensional granular flows in two simple setups: the shear plane and the Poiseuille configurations. We will also restrict ourselves to the cases of steady and fully developed flows, *i.e.* flows whose properties depend neither on time nor on the position along the main flow axis.

The outline of this article is the following. In the next section we will present the assumptions used in this work. Section 3 is devoted to the presentation of the description of the analytical resolution that we used. Then, we will present the analytical approach, results and discussions for the shear plane flows (Sect. 4) and the Poiseuille flows (Sect. 5). Finally we will present our conclusions.

2 Simplifying assumptions: steady and fully developed flows

The analytical resolution of the Navier Stokes equations in the case of Newtonian fluids is difficult. The $\mu(I)$ rheology introduces a non-constant viscosity that complicate the resolution further. To bypass these difficulties, we restrict ourselves to the case of steady and fully developed granular flows.

Let us define the x -axis as the horizontal axis from left to right and the y -axis as the vertical axis from bottom to top. The used assumptions are:

- the flow is steady i.e. it no longer depends on time, which implies that $\partial \mathbf{u} / \partial t = 0$.
- the flow is fully developed; that is, its properties (e.g. velocity) are nearly invariant along the main direction of flow. Consequently, we have for velocity, $\partial \mathbf{u} / \partial x = 0$ (flow does not depend on x -direction), and then the y component of the velocity is equal to zero, $u_y = 0$.
- the pressure P is supposed to be hydrostatic within the flow i.e. $P = P_0 + \rho g(H - y)$ where H is the height of the flow and P_0 the external pressure. Note that this hypothesis was tested many times by simulation of discrete elements in different geometries [29, 30].

Taking these assumptions into account, the system to solve (5) is reduced to the following differential equation:

$$\frac{\partial}{\partial y} \left(\eta(y) \frac{\partial u}{\partial y} \right) = K, \quad (7)$$

where $K = \partial P / \partial x$ is the pressure gradient in the direction of flow (assumed to be constant) and $\eta(y) = \mu(I)P / |\dot{\gamma}|$ is the effective dynamic viscosity. Let us recall here that the variations of the packing fraction are neglected ($\Phi = 0.6$ uniformly within the system).

3 Analytical resolution

We solve analytically the nonlinear Eq. (7). The steps of this calculation are the followings:

1. We integrate analytically with respect to y the pressure gradient $\partial P / \partial x$ (assumed to be constant). This leads to:

$$\eta(y) \frac{\partial u}{\partial y} = K y + k_1, \quad (8)$$

where k_1 is the constant of integration.

2. We solve directly the Eq. (8) with $\dot{\gamma} = \partial u / \partial y$ as unknown. Let us recall here that $\eta(y) = \mu(I)P / |\dot{\gamma}|$ is the effective viscosity.
3. We integrate with respect to y the result $\partial u / \partial y$ and call k_2 the constant of integration.
4. By applying the boundary conditions of the studied configuration, we obtain a system of two equations which allows the determination of the unknowns k_1 and k_2 .

As mentioned above, in the following two sections, we will apply this resolution to two simple configurations: the plane shear flow (Sect. 4) and the Poiseuille flow (Sect. 5).

4 Shear plane flow

We applied the resolution described in Sect. 3 to the shear plane configuration (see Fig. 1) with no pressure gradient ($K = 0$). The granular medium is located

between two plates separated by a height H , P_0 is the pressure resulting from a vertical stress on the top plate, which moves at a constant velocity U_0 , the lower one being fixed. This geometry has been studied intensively both experimentally [3] and numerically by discrete element methods [3, 4, 29].

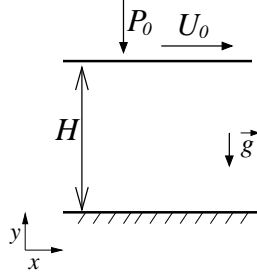


Fig. 1 Sketch of the 2D shear plane configuration. H is the distance between the two plates, P_0 is the pressure resulting from a vertical stress on the top plate and U_0 the horizontal velocity of the same plate.

4.1 Dimensionless formulation

In Eq. (7), we have seven parameters that characterize the flow: U_0 , P_0 , H , ρ , ρ_s , g and d which are respectively the velocity of the top plate, the pressure on the top plate, the height between the plates, the density of the granular medium, the grain density, the gravity and the diameter of the grains. To write Eq. (7) in a dimensionless form, we must choose three scales: a length scale H , a velocity scale U_0 and a pressure scale P_0 . The dimensionless variables are then: $y^* = y/H$, $u^* = u/U_0$ and $P^* = P/P_0$. From Vaschy-Buckingham theorem [31] we can then reduce our set of parameters to only four dimensionless ones:

$$\Lambda = \frac{H}{d}, \quad \varepsilon = \frac{\rho g d}{P_0}, \quad \alpha = \frac{U_0}{\sqrt{P_0/\rho_s}} \quad \text{and} \quad \Phi = \frac{\rho}{\rho_s}.$$

The variables which depend on y^* are: $P^*(y^*) = 1 + \varepsilon \Lambda (1 - y^*)$, $I(y^*) = \frac{\alpha |\dot{\gamma}^*|}{\Lambda \sqrt{P^*(y^*)}}$, and $\mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_0/I + 1}$. Note that μ_s , μ_2 and I_0 are not considered as variables because they are constants of the $\mu(I)$ -rheology. Thus, the dimensionless equation to solve is:

$$\frac{\partial}{\partial y^*} \left[|\dot{\gamma}^*| \left(\varepsilon \Lambda (1 - y^*) + 1 \right) \left(\frac{\alpha (\mu_2 - \mu_s)}{\Lambda I_0 \sqrt{\varepsilon \Lambda (1 - y^*) + 1}} + \frac{\mu_s}{|\dot{\gamma}^*|} \right) \right] = 0. \quad (9)$$

In order to easily solve this equation, we must get rid of the absolute value that applies to the shear rate $\dot{\gamma}^*$. In our geometry, the top plate moves at a positive velocity and the bottom one is motionless. The shear rate within the granular system is therefore positive, or equal to zero:

$$\dot{\gamma}^* \geq 0 \quad \text{then} \quad |\dot{\gamma}^*| = \dot{\gamma}^*. \quad (10)$$

This allows us to obtain the following expression for the shear rate

$$\dot{\gamma}^* = - \frac{\Lambda I_0 \sqrt{\varepsilon \Lambda (1 - y^*) + 1} (\varepsilon \Lambda \mu_s y^* - \varepsilon \Lambda \mu_s - \mu_s + k_1)}{\alpha (\varepsilon \Lambda \mu_2 y^* - \varepsilon \Lambda \mu_2 - \mu_2 + k_1)}. \quad (11)$$

where k_1 is the constant of integration of Eq. (9).

The velocity is then obtained by integrating the shear rate with respect to y^* :

$$u^*(y^*) = k_2 + \frac{k_1^{3/2} I_0 (\mu_s - \mu_2)}{\alpha \varepsilon \mu_2^{5/2}} \log \left(\frac{2 \mu_2 \sqrt{\varepsilon \Lambda (1 - y^*) + 1} - 2 \sqrt{\mu_2 k_1}}{2 \mu_2 \sqrt{\varepsilon \Lambda (1 - y^*) + 1} + 2 \sqrt{\mu_2 k_1}} \right) + \quad (12)$$

$$\frac{2 I_0}{3 \alpha \varepsilon \mu_2^2} \left(\mu_2 \mu_s (\varepsilon \Lambda (1 - y^*) + 1)^{3/2} + 3 k_1 (\mu_s - \mu_2) \sqrt{\varepsilon \Lambda (1 - y^*) + 1} \right)$$

To completely define the velocity profile, it is necessary to determine the constants k_1 and k_2 . This is done in the next section through the use of the boundary conditions.

4.2 Boundary conditions

Although the use of the $\mu(I)$ rheology close to boundaries, where the influence of the fluctuating energy flux may not be disregarded, is questionable we assume here that such an approximation does not modify significantly the features of the flow. The relevancy of that assumption will be discussed in the last section of the present paper. Assuming that there is no slip on the walls, the velocity of the granular material at $y^* = 1$ is equal to that of the top plate i.e. U_0 . Since the pressure P^* decreases with increasing y^* , we may observe situations where the yield criterion $\tau/P > \mu_s$ is verified only if y^* is larger than a critical value $y_{critical}^*$. Therefore, two situations have to be considered. First, the case where the yield criterion is verified at any depth. In that case, the velocity of the granular medium at the bottom plate is equal to the one of the bottom plate, i.e. zero. The corresponding boundary condition is therefore $u^*(y^* = 0) = 0$. The other situation is the case where the yield stress condition is only satisfied for $y^* \geq y_{critical}^*$. The flow is then localized close to the moving plate between $y^* = 1$ and $y_{critical}^*$. In such a case, the former boundary condition is still valid but the latter has to be replaced by $\tau(y^* = y_{critical}^*)/P^*(y^* = y_{critical}^*) = \mu_s$ and by $u^*(y^* = y_{critical}^*) = 0$. From a practical point of view, $y_{critical}^*$ depends on α and ε , dependence which will be studied in the following. The flow localization is therefore observed only if the dimensionless height of the channel Λ is greater than the dimensionless length $\Lambda_{critical} = \Lambda(1 - y_{critical}^*)$. Note that, for given ε and α , if Λ is set equal to $\Lambda_{critical}$, the conditions $\tau(y^* = y_{critical}^*)/P^*(y^* = \Lambda_{critical}) = \mu_s$ and $u^*(y^* = y_{critical}^*) = 0$ are equivalent to $u^*(y^* = 0)$.

If the yield criterion is satisfied at any depth, the two boundary conditions allow us to find the values of k_1 and k_2 in step 4 of the resolution (see Sect. 3). To solve this nonlinear equation, we use the second boundary condition and Eq. (12) to

write k_2 as a function of k_1 . Then, the same equation and the other boundary condition are used to get the value of k_1 by using Newton's iterative method. Note that it is necessary to choose an adequate initial value of k_1 . Indeed, if we set $f(k_1) = 0$ the equation to be solved, the graph of $f(k_1)$ has a vertical asymptote, which correspond to $I \rightarrow +\infty$, at the point of abscissa $k_1 = \mu_2$, and no real values for $k_1 > \mu_2$. Practically, we therefore choose $k_{1\text{init}} = \mu_2 - 10^{-4}$.

If the flow is localized between $y^* = 1$ and $y^* = y_{\text{critical}}^*$ the system composed of the three boundary conditions is solved numerically by using Newton's iterative method.

4.3 Results

We have previously shown that the description of the flow depends on four parameters Λ , ε , α and Φ . As mentioned above, the variations of the packing fraction Φ are neglected within the flow, so we restrict ourselves to the study of the influence of the other three parameters. Note that by definition, these parameters are all positive and different to zero except ε which can be zero if gravity is not taken into account. In that case, the shear rate $|\dot{\gamma}^*|$ (see Eq. (11)) reduces to a constant

$|\dot{\gamma}^*| = -\frac{\Lambda I_0 (k_1 - \mu_s)}{\alpha (k_1 - \mu_2)}$ and the velocity profile becomes linear with y , as follows:

$$u^*(y^*) = -\frac{\Lambda I_0 (k_1 - \mu_s) y^*}{\alpha (k_1 - \mu_2)} + k_2.$$

In the general case ($\varepsilon \neq 0$), we assigned values to the variables ε and α , and vary Λ . Experimentally that corresponds to a variation of the height between the plates, or a variation of the grain diameter. We chose values of α and ε compatible with typical experimental situations on glass beads. Thus, by choosing $\rho = 1.5 \times 10^3 \text{ kg/m}^3$, $g = 9.81 \text{ m/s}^2$, $d = 0.5 \times 10^{-3} \text{ m}$, $P_0 = 1,000 \text{ Pa}$ and $U_0 = 100 \text{ mm/s}$ we obtain $\varepsilon = 0.15$ and $\alpha = 0.007$.

Figure 2 shows the velocity profiles for different values of Λ . Note that for very small values of Λ ($\Lambda \leq 2$), the velocity profile tends to be linear, whereas for larger values ($\Lambda = 5$ and 10) the velocity profile is more curved. If the gap between plates are even greater i.e. $\Lambda > \Lambda_{\text{critical}}$, the yield criterion is not satisfied between $y^* = 0$ and $y^* = y_{\text{critical}}^*$ leading to a localization of the flow between the latter depth and $y^* = 1$. Therefore, as expected, three cases can be observed:

- for $\Lambda = \Lambda_{\text{critical}}$ the granular system flows at any height and the shear rate is equal to zero at the bottom plate,
- for $\Lambda < \Lambda_{\text{critical}}$ the flow also occurs at any height but the shear rate is strictly positive at the bottom plate,
- for $\Lambda > \Lambda_{\text{critical}}$ the flow is localized. The static zone corresponds to $y^* \in [0, 1 - \Lambda_{\text{critical}}/\Lambda]$ and the flowing zone to $y^* \in [1 - \Lambda_{\text{critical}}/\Lambda, 1]$.

Note that, practically, the value of $\Lambda_{\text{critical}}$ is determined numerically by dichotomy on a given interval of Λ . Interestingly, as long as $\Lambda_{\text{critical}}$ is defined (i.e. $\Lambda \geq \Lambda_{\text{critical}}$), it does not depend on Λ . This result is in agreement with experiments [3, 32, 33] as well as discrete element simulations [4, 29] that report that under some conditions the flow in a plane shear cell is localized close to the moving surface. Below the aforementioned shear layers, the system is quasistatic. An

increase of the height of the system modifies neither the height of the shear layer nor the velocities of its grains. However, it should be noted that considering the zone below the shear layer as a purely static area is an approximation. Although it has been used many times [32, 34–38] it does not reflect the reality: the grains actually move intermittently [39, 40] and the corresponding average profile decreases exponentially with depth. This discrepancy comes from the limitation of the $\mu(I)$ -rheology that is not able to take into account the non-local effects responsible of the aforementioned intermittent motion.

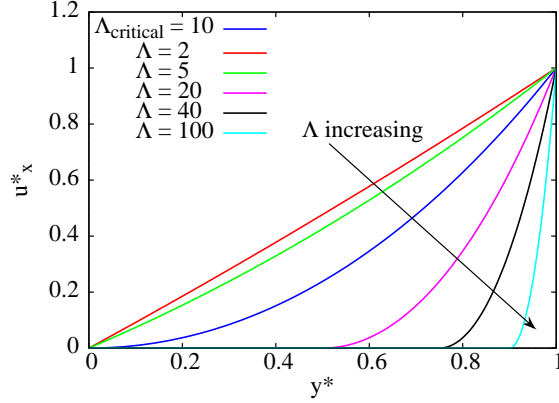


Fig. 2 Velocity profiles obtained by varying Λ for $\varepsilon = 0.15$ and $\alpha = 0.007$. For $\Lambda = \Lambda_{\text{critical}}$, the shear rate is exactly zero at $y^* = 0$. For $\Lambda > \Lambda_{\text{critical}}$ there is an area for which the velocity is equal to zero.

4.4 Discussion

4.4.1 Proposed law for $\Lambda_{\text{critical}}$

We will now investigate the dependency of $\Lambda_{\text{critical}}$ with the other two parameters: ε and α . Figure 3 reports the variations of $\Lambda_{\text{critical}}$ versus α ($\alpha \in [0.01, 1]$) and for different values of ε . It shows that

$$\Lambda_{\text{critical}} \approx f(\varepsilon) \alpha^{\frac{1}{2}}, \quad (13)$$

where $f(\varepsilon)$ is a function that describes the dependence of $\Lambda_{\text{critical}}$ with ε . Figure 4 reports the variations of Λ for $1/\varepsilon$ ranged from 0 to 100 and for different values of α . Let us recall that $1/\varepsilon$ is proportional to the pressure P_0 . Therefore, studying the influence of $1/\varepsilon$ is equivalent to studying the effect of the external pressure P_0 . We observe that

$$\Lambda_{\text{critical}} \approx g(\alpha) 1/\varepsilon^{\frac{1}{2}}, \quad (14)$$

where $g(\alpha)$ is a function that describes the dependence of $\Lambda_{\text{critical}}$ with α . From equations (13) and (14) we deduce:

$$\Lambda_{\text{critical}} \approx A \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}, \quad (15)$$

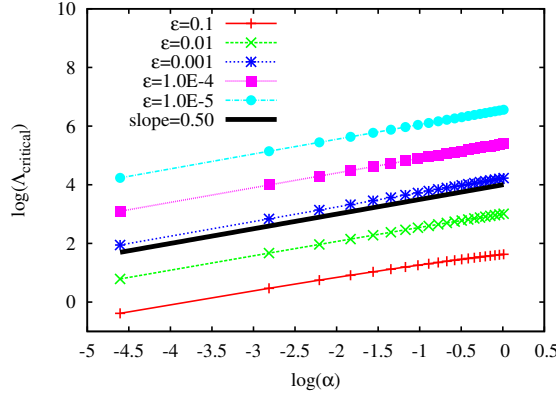


Fig. 3 Critical height $\Lambda_{\text{critical}}$ as a function of the dimensionless parameter α for several values of ε . For a given ε , Λ is proportional to $\alpha^{1/2}$. \log refers to the neperian logarithm.

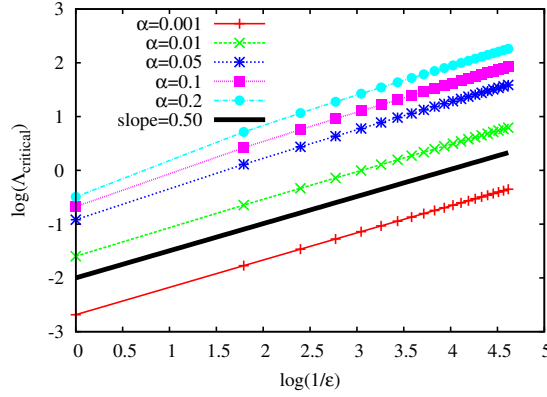


Fig. 4 Critical height $\Lambda_{\text{critical}}$ versus the dimensionless parameter $1/\varepsilon$ for several values of α . For a given α , Λ is proportional to $1/\varepsilon^{1/2}$. \log refers to the neperian logarithm.

where $A = 2.2$ is calculated at point $\alpha = 10^{-2}$, $\varepsilon = 10^{-2}$.

An important point should be stressed out. In Figs. 3 and 4, $\Lambda_{\text{critical}}$ takes any value between ≈ 0 and ≈ 10 . However, since the height of a granular system cannot be smaller than the diameter of a grains, we have $H > d$ i.e. $\Lambda > 1$. The latter condition and relation (15) lead to the following condition $2.2\sqrt{\alpha} \geq \sqrt{\varepsilon}$. Figure 5 shows the interval of validity of the approached law given by Eq. (15). We can observe that the aforementioned simplified equation does not hold for large values of ε and α (for $\log_{10}(\varepsilon^{-1/2}) < 0.5$ and $\log_{10}(\alpha^{1/2}) > -0.5$). On the other side, the approached law better fits the exact results for small values of $1/\varepsilon$ and α . That approached law as well as the relative deviations from that law – whose isovalues are given by Fig. 6 – will be discussed in next section.

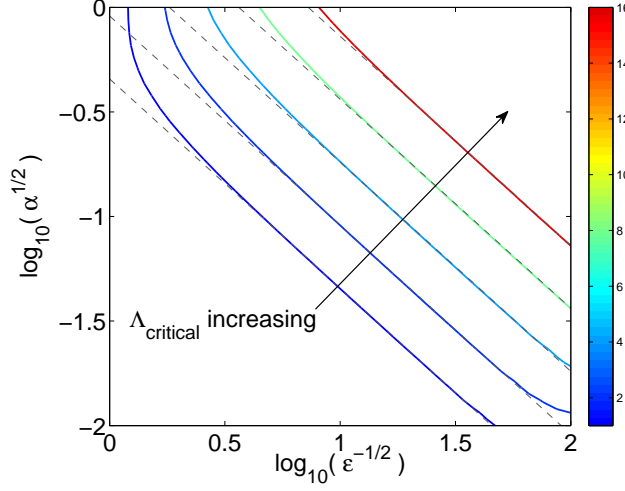


Fig. 5 Isovalue lines of $\Lambda_{\text{critical}} = 1, 2, 4, 8, 16$ (continuous line), compared to the approximated law (15) (dashed line). The approximated law fits well the isovalue lines for values of α and $1/\varepsilon$ near to $\alpha = 10^{-2}$, $\varepsilon = 10^{-2}$. The map was obtained by a plot of isovalues of a matrix of 34 by 34 points.

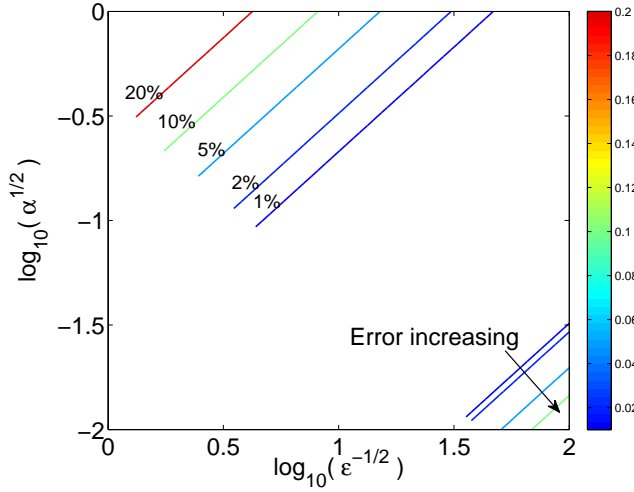


Fig. 6 Isovalue lines of the relative error= 20%, 10%, 5%, 2%, 1% of approximate law (15). The map was obtained by a plot of isovalues of a matrix of 34 by 34 points.

4.4.2 Justification of the simplified law for $\Lambda_{\text{critical}}$

In order to justify the dependency of $\Lambda_{\text{critical}}$ in $\sqrt{\alpha/\varepsilon}$, we will derive below this relationship by a simplified approach. That will allow us to better understand the origin of its domain of validity (see Fig. 5). In the configuration of the shear plane, the horizontal pressure gradient is zero $\partial P(y)/\partial x = 0$, the equation of motion

therefore is $\partial\tau(x, y)/\partial y = 0$, i.e.

$$\frac{\partial\tau}{\partial y} = \frac{\partial\mu(y)P(y)}{\partial y} = 0. \quad (16)$$

After integration, we obtain

$$\mu(y) = \frac{A}{P(y)}, \quad (17)$$

where A is a constant.

One of the simplifying assumptions adopted (see Sect. 2) is the hydrostatic character of the pressure within the granular medium. Thus, in dimensionless form, we have $P^*(y) = 1 + \varepsilon\Lambda(1 - y^*)$. Then, we can write the Eq. (16) as

$$\mu(y^*) = \frac{A}{1 + \varepsilon\Lambda(1 - y^*)}. \quad (18)$$

In the following, we will consider the case where the flow height is exactly equal to the critical height for which the shear stress becomes zero at the bottom plate (at $y^* = 0$). In this case, at this same plate, the granular system is at the limit of the static state and, consequently, we have $\mu(y^* = 0) = \mu_s$. Using this last relation in Eq. (18) we can find the expression of the constant A and obtain for $\mu(y^*)$ the following expression

$$\mu(y^*) = \mu_s \frac{1 + \varepsilon\Lambda_{\text{critical}}}{1 + \varepsilon\Lambda_{\text{critical}}(1 - y^*)}. \quad (19)$$

At $y^* = 1$ this equation becomes

$$\mu(y^* = 1) = \mu_s(1 + \varepsilon\Lambda_{\text{critical}}). \quad (20)$$

We have an expression for the critical height $\Lambda_{\text{critical}}$ depending on the effective friction coefficient $\mu(y^*)$. It is then sufficient to express the coefficient of friction as a function of α and ε to derive an expression for $\Lambda_{\text{critical}}$ in terms of these quantities. This can be done with the empirical formula connecting the effective friction with the inertial number I : $\mu(I) = \mu_s + (\mu_2 - \mu_s)/(1 + I_0/I)$. The inertial number depends on y through pressure and shear rate: $I(y^*) = \dot{\gamma}(y^*)\alpha/(\Lambda_{\text{critical}}\sqrt{P^*(y^*)})$.

The pressure dependency with y is known (hydrostatic assumption) contrary to that of the shear rate. To overcome this lack, we can assume that the velocity profile is linear between $y^* = 1$ and $y^* = 0$. In doing so, we underestimate the shear rate at the surface $y^* = 1$ but it seems reasonable to assume that this approximation does not alter the dependency of $\dot{\gamma}$ in respect to α and ε . Thus, within this approximation, the shear rate at the upper plate is $\dot{\gamma} \approx U_0/H_{\text{critical}}$ which corresponds to a dimensionless shear $\dot{\gamma}^*$ of the order 1. Moreover, still at $y^* = 1$, we have $P^*(y^* = 1) = 1$ and $I \approx \alpha/\Lambda_{\text{critical}}$. We can then deduce the following expression for $\mu(y^*)$:

$$\mu(y^* = 1) \approx \mu_s + (\mu_2 - \mu_s) \frac{\alpha}{\alpha + I_0\Lambda_{\text{critical}}}.$$

Then, this equation can be used to substitute $\mu(y^* = 1)$ in the Eq. (19) for which $y^* = 1$ leading to the following second-order equation:

$$\Lambda_{\text{critical}}^2 + \frac{\alpha}{I_0}\Lambda_{\text{critical}} - \frac{\mu_2 - \mu_s}{I_0\mu_s} \frac{\alpha}{\varepsilon} = 0, \quad (21)$$

which has real solutions only if

$$\alpha\varepsilon \geq -4 \left(\frac{\mu_2 - \mu_s}{\mu_s} \right). \quad (22)$$

Since the left hand side of this equation is negative, and the quantities α and ε are positive, this condition is always satisfied. The only positive solution, physically acceptable, is therefore

$$\Lambda_{\text{critical}} = \sqrt{\frac{\alpha}{\varepsilon} \frac{\mu_2 - \mu_s}{\mu_s I_0}} \left(\sqrt{1 + \frac{\alpha\varepsilon\mu_s}{4I_0(\mu_2 - \mu_s)}} - \sqrt{\frac{\alpha\varepsilon\mu_s}{4I_0(\mu_2 - \mu_s)}} \right) \quad (23)$$

The dependency $\Lambda_{\text{critical}} \propto \sqrt{\alpha/\varepsilon}$ is found if $\zeta = \alpha\varepsilon\mu_s/[4I_0(\mu_2 - \mu_s)] \ll 1$. This condition corresponds to neglect the first order term in $\Lambda_{\text{critical}}$ in Eq. (21). In this case, we have

$$\Lambda_{\text{critical}} = \sqrt{\frac{\mu_2 - \mu_s}{\mu_s I_0}} \sqrt{\frac{\alpha}{\varepsilon}} \approx 1.57 \sqrt{\frac{\alpha}{\varepsilon}}. \quad (24)$$

Note the proximity of the coefficient 1.57 with the coefficient 2.2 of Eq. (15). The weak difference comes from the approximation made above on the shear rate at the upper plate for $\Lambda = \Lambda_{\text{critical}}$. By estimating the shear rate more accurately by using the Fig. 2 we find the right factor.

As mentioned above the law $\Lambda_{\text{critical}} = 2.2\sqrt{\alpha/\varepsilon}$ is an approximation and the isovalue lines of the relative error are reported in Fig. 6. The simple justification mentioned above is also able to explain the shape of those isovalues lines. Indeed, to obtain such a simple relation between $\Lambda_{\text{critical}}$, α and ε we have to assume that ζ is negligible with respect to 1. If it is not the case, but if ζ is small, we can perform a first order Taylor expansion in ζ of Eq. (23) that leads to

$$\Lambda_{\text{critical}} \approx \sqrt{\frac{\alpha}{\varepsilon} \frac{\mu_2 - \mu_s}{\mu_s I_0}} \left(1 - \frac{\sqrt{\alpha\varepsilon}}{2\sqrt{I_0}} \sqrt{\frac{\mu_s}{\mu_2 - \mu_s}} \right).$$

Therefore, the relative error can be approximated by

$$|\Delta\Lambda_{\text{critical}}/\Lambda_{\text{critical}}| \approx \frac{\sqrt{\alpha\varepsilon}}{2\sqrt{I_0}} \sqrt{\frac{\mu_s}{\mu_2 - \mu_s}},$$

justifying why the isovalues are more or less straight lines when they are plotted in the plane $(\log \alpha^{1/2}, \log \varepsilon^{-1/2})$.

5 Poiseuille flow

The second configuration for which we applied the semi-analytical resolution is the Poiseuille flow. The granular medium flows in a channel, i.e. between two stationary plates (see Fig. 7) and a pressure difference between inlet and outlet of the channel is imposed. In this configuration we have been working in the absence of gravity. Here again we assume that the packing fraction is uniform within the flow and equal to 0.6.

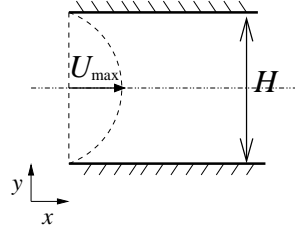


Fig. 7 Sketch of the 2D Poiseuille configuration. H is the distance between the two plates and U_{max} the maximum velocity of the imposed parabolic profile as input.

5.1 Dimensionless formulation

Following the same way as in the case of shear plane, to solve the Eq. (7), we have six parameters that characterize our flow, i.e. K [the pressure gradient, see Eq. (8)], P_0 , H , ρ , ρ_s and d . Let us recall that gravity is not taken into account. To write Eq. (7) into a dimensionless form, we give three scales: a length scale H , a velocity scale $\sqrt{-K \frac{H}{\rho_s}}$ and a pressure scale P_0 . The rescaled variables are then

written as follows: $y^* = \frac{y}{H}$, $u^* = \frac{u}{\sqrt{-K \frac{H}{\rho_s}}}$, $P^* = \frac{P}{P_0}$. This allows us to have three

dimensionless variables:

$$\Lambda = \frac{H}{d}, \quad \beta = \frac{\sqrt{-K H}}{\sqrt{P_0}} \quad \text{and} \quad \Phi = \frac{\rho}{\rho_s}.$$

The variables which depends on y^* are: $P^*(y^*) = 1$, $I(y^*) = \frac{\beta |\dot{\gamma}^*|}{\Lambda \sqrt{P^*(y^*)}}$, and

$$\mu(I) = \mu_s + \frac{\mu_2 - \mu_s}{I_0/I + 1}.$$

In the Poiseuille configuration we have a plane of symmetry (Fig. 7) which allows us to restrict our study to the half of the domain ($y^* \in [0, 1/2]$) where the shear rate is positive $\dot{\gamma}^* \geq 0$ then $|\dot{\gamma}^*| = \dot{\gamma}^*$.

Thus the system is solved in dimensionless form:

$$\frac{\partial}{\partial y^*} \left(\frac{\beta |\dot{\gamma}^*| (\mu_2 - \mu_s)}{\Lambda I_0 + \beta |\dot{\gamma}^*|} + \mu_s \right) = -\beta^2, \quad (25)$$

with

$$|\dot{\gamma}^*| = -\frac{\Lambda I_0 (\beta^2 y^* + \mu_s - k_1)}{\beta (\beta^2 y^* + \mu_2 - k_1)}, \quad (26)$$

where k_1 is the constant of integration. By integrating this equation with respect to y^* , we obtain, if the yield criterion is satisfied, the following expression of the velocity:

$$u^*(y^*) = k_2 - \frac{\Lambda I_0}{\beta^3} \left(y^* \beta^2 + (\mu_s - \mu_2) \log (\beta^2 y^* + \mu_2 - k_1) \right), \quad (27)$$

where k_2 is the constant of integration.

5.2 Boundary conditions

The two boundary conditions that we have in the half-Poiseuille configuration are:

- the shear stress is equal to zero at the center line, i. e. $\tau(y^* = 1/2) = 0$.
- we consider that the granular medium does not slide at the plate, i.e. $u^*(y^*) = 0$ at $y^* = 0$.

These two conditions allow us to find the value of k_1 and k_2 in step 4 of resolution (see Sect. 3), thus:

$$k_1 = \beta^2/2,$$

and

$$k_2 = -\frac{\Lambda I_0 (\mu_2 - \mu_s)}{\beta^3} \log\left(\mu_2 - \frac{\beta^2}{2}\right).$$

The dimensionless shear stress is then given by $\tau^* = \beta^2(1/2 - y^*)$. Note that, since this quantity has an upper boundary μ_2 , a steady state cannot be attained, in the framework of this theory, if β (the dimensionless pressure gradient) is higher than $\sqrt{2\mu_2}$. In the following we will therefore consider that $\beta < \sqrt{2\mu_2}$. Let us now determine y_c^* , the value of y^* for which the yield criterion is no more satisfied i.e. $\tau < \mu_s P$. From the latter expression of τ it corresponds to $y_c^* = 1/2 - \mu_s/\beta^2$. Below this value, the yield criterion is satisfied and the system is sheared. Above, it behaves like a plug flow. Physically, y_c^* cannot be lower than 0. This conditions leads to $\beta^2 > 2\mu_s$. So, in the following, we will consider that $\beta \in [\sqrt{2\mu_s}, \sqrt{2\mu_2}]$. After integration of the shear rate, we obtain for $y^* < y_c^*$, the following expression of the velocity:

$$u^*(y^*) = -\frac{\Lambda I_0}{\beta^3} \left(\beta^2 y^* + (\mu_s - \mu_2) \log\left(-\frac{2\beta^2 y^* + 2\mu_2 - \beta^2}{\beta^2 - 2\mu_2}\right) \right). \quad (28)$$

The maximum value of the velocity is then

$$u_m^* = u^*(y^* = 1/2 - \mu_s/\beta^2) = -\frac{\Lambda I_0}{\beta^3} \left(\beta^2/2 - \mu_s + (\mu_s - \mu_2) \log\left(\frac{\mu_2 - \mu_s}{\mu_2 - \beta^2/2}\right) \right). \quad (29)$$

Interestingly, the limit case $y_c^* = 1/2$ is only obtained for infinite pressure gradient i.e. $\beta \rightarrow +\infty$ which is incompatible with the aforementioned condition $\beta \in [\sqrt{2\mu_s}, \sqrt{2\mu_2}]$. Therefore, in such a geometry, the flow always displays a plug flow at the center of the cell. Its minimum and maximum sizes are respectively μ_s/μ_2 (obtained for $\beta \rightarrow \sqrt{2\mu_2}$) and 1 (obtained for $\beta \rightarrow \sqrt{2\mu_s}$).

5.3 Results and Discussion

In the current configuration, the description of the flow depends on two parameters Λ and β as the variations of Φ are neglected ($\Phi = 0.6$). For a granular flow with the parameters $\rho = 1.5 \times 10^3 \text{ kg/m}^3$, $g = 9.81 \text{ m/s}^2$, $d = 0.5 \times 10^{-3} \text{ m}$, $H = 0.1 \text{ m}$, $P_0 = 100 \text{ Pa}$ and $K = -100 \text{ Pa/m}$ we have $\Lambda = 20$ and $\beta = 0.31$. We study the influence of these parameters on the velocity profile of the flow. Equation (28) clearly shows that the amplitude of the velocity profile is proportional to Λ . Thus, the study will be restricted to the influence of the parameter β (Λ is kept constant

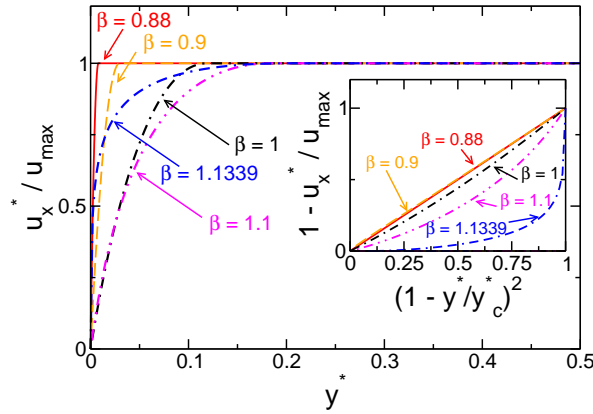


Fig. 8 Variation of velocity profile for different β and for $\Lambda = 20$. The Inset represents the profile of the sheared regions which tend to be parabolic for $\beta \rightarrow \sqrt{2\mu_s} = 0.8739$.

and equal to 20).

Figure 8 reports the profiles obtained for different values of β . As expected a plug flow is visible at the center of the channel for any values of $\beta \in [\sqrt{2\mu_s}, \sqrt{2\mu_2}]$. Those profiles are somewhat close to the ones obtained by the “Da Vinci fluid” model described in [41]. They also look similar to the profiles obtained with Bingham plastic fluids, i.e. a material that behaves as a rigid body at low stresses but flows as a viscous fluid at high stress. Let us recall here that Poiseuille flows of such fluids display a plug at the center of the cell and a parabolic velocity profile close to the sidewalls. To quantify this resemblance, we report in the inset of Fig. 8, the quantity $(1 - u_x^*/u_{max}^*)$ versus $(1 - y^*/y_c^*)^2$. For Bingham plastic fluids, those two dimensionless quantities are equal. Interestingly, we find that the velocity profiles correspond to those of a Bingham plastic fluid for small values of β (e.g. $\beta = 0.88$, and 0.90). They flatten for larger values (e.g. $\beta = 1$, 1.1 , 1.339). Another quantity of interest is the maximum velocity u_m^* , which is equal to $u^*(y^* = y_c^*)$. Figure 9, which reports previous quantity as a function of β , shows that u_m^* is equal to zero for $\beta = \sqrt{2\mu_s}$ and diverges when β tends to $\sqrt{2\mu_2}$. The presence of this divergence can be explained as follows. The balance of flow momentum over half the cell width gives that the pressure gradient K times the half width $H/2$ is balanced by the difference in shear stresses on the wall and on the center of the cell. The former shear stress is $[\mu_s + (\mu_2 - \mu_s)/(1 + I_0/I)]P_0$ and the latter is equal to zero. Consequently, $KH/2$ can increase no further than the limiting stress difference $\mu_2 P_0$ obtained as I and u_m^* become infinite.

This figure also shows that for small value of β ($\beta < 0.95$), the velocity U_{max} is the same as in the case of a Bingham plastic fluid (i.e. a parabolic profile between 0 and y_c^* , then a plug flow) although the viscosity of the granular fluid is not that of a Bingham plastic fluid. Figure 8 also shows that for the same range of β ($\beta < 1$) the velocity profile of the sheared region is approximated by a parabola. To justify Bingham-like behavior for $\beta \rightarrow \sqrt{2\mu_s}$, let us first recall that for such fluid, when the yield criterion is satisfied, the shear stress is equal to $\tau_c + \eta_n \dot{\gamma}$, where τ_c is the yield stress and η_n a constant Newtonian viscosity. Then, let us consider

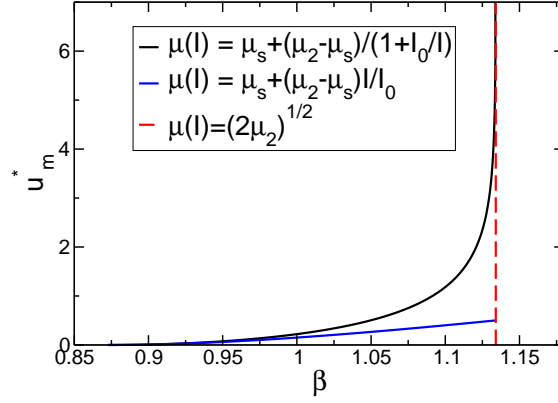


Fig. 9 Variation of u_m^* as a function of β and for $\Lambda = 20$. A Bingham plastic fluid and a granular material obeying the $\mu(I)$ -rheology display a similar behavior when $\beta \rightarrow 0$. On the contrary, the latter displays a divergence in velocity for when $\beta \rightarrow \sqrt{2\mu_2} = 1.1344$ whereas the former remains linear.

now the $\mu(I)$ -rheology and assume that I is much smaller than I_0 , justifying the approximation $\mu(I) = \mu_s + (\mu_2 - \mu_s)I/I_0$. If the yield criterion is satisfied, the shear stress is then Bingham-like:

$$\tau = \tau_c + \eta_n |\dot{\gamma}|, \quad (30)$$

with $\tau_c = \mu_s P$ and $\eta_n = (\mu_2 - \mu_s)\sqrt{\rho_S \bar{P}}/I_0$. In such a case, we obtain the following expressions of the velocity for $y^* \leq y_c^*$:

$$u^* = \frac{I_0 \Lambda}{2\beta(\mu_s - \mu_2)} \left(\beta^2 y^{*2} + (2\mu_s - \beta^2)y^* \right) \quad (31)$$

Its maximum value is given by

$$u_m^* = u^*(y^* = y_c^*) = \frac{\Lambda I_0}{8\beta^3} \frac{(\beta^2 - 2\mu_s)^2}{\mu_s - \mu_2}. \quad (32)$$

It is worth noting that, if the expression used for the effective friction coefficient $\mu(I)$ is the simplified one (Eq. 4), β has no upper limit. Therefore, the position of the plug flow y_c^* belongs to the range $[0, 1/2]$. Then, when $\beta \rightarrow \infty$ the half-length of the plug flow, λ_{plug} , tends to zero and the system is sheared all along its length. This point is illustrated on Fig. 10 where λ_{plug} is reported versus β . The gray zone corresponds to the ranges of β and λ_{plug} that can be reached using the full expression of $\mu(I)$ (Eq. 2). If the simplified expression is used (Eq. 4), the values of λ_{plug} are bounded between 0 and 1 and those of β between $\sqrt{2\mu_s}$ and $+\infty$. From the two latter equations we can easily show that, in such a condition, the velocity profile tends toward a parabola and the maximum velocity diverges like β^{-1} .

6 Conclusion and discussion

In this paper, we studied theoretically granular flows in the framework of the $\mu(I)$ rheology. We focused on steady and fully developed granular flows in two

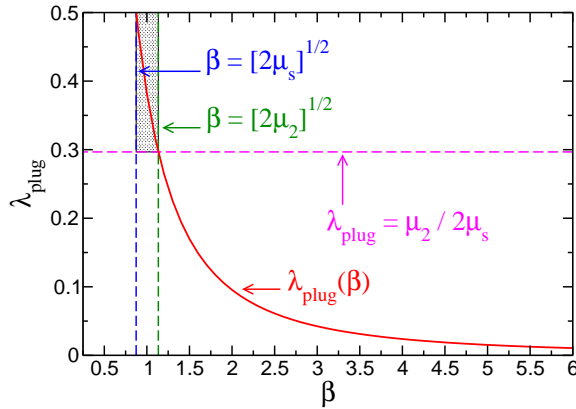


Fig. 10 The half-length of the plug flow in the Poiseuille configuration depends on the gradient pressure β . Depending on the expression used for $\mu(I)$ (see text for details), this half-length is between 0.5 and $\mu_s/2\mu_2$ (full expression given by Eq. 2 - gray zone) or 0.5 and 0 (simplified expression given by Eq. 4).

geometries: the shear plane and Poiseuille. We obtained results can be summarized as follow:

In the shear plane configuration, we have shown that for appropriate parameters, the flow is spatially localized. This is consistent with many experimental observations. We have also identified a law characterizing the flow, including $\Lambda_{\text{critical}} \propto \sqrt{\alpha/\varepsilon}$ [see Eqs. (15) and (24)], i.e. the height H on which the granular medium is in motion is proportional to $U_0^{1/2} P_0^{1/4}$. Although this law is not valid for all values of α and ε the domain of applicability seems very broad.

In the Poiseuille configuration, we have described in detail the influence of the parameter β which is a function of the pressure gradient in the flow. We have shown that the granular material flows only if the pressure gradient is greater than a threshold value and that, under certain circumstances, the system behaves like a Bingham plastic fluid.

As mentioned above, the $\mu(I)$ rheology does not take into account the influence of the fluctuating energy flux that cannot be disregarded close to a boundary, especially when the granular material is not dense. In the case of the plane shear flow, we obtain a qualitative agreement with experiments [3] and simulations [29, 40] which suggests that neglecting the energy flux is a reasonable assumption. The case of Poiseuille flow is more difficult since few experimental data are available in such a configuration. Our goal was not to compare directly experimental results with the $\mu(I)$ rheology but to apply the latter to a geometry commonly used in fluid mechanics. However the following discussion will shed some light on that particular point.

In this work, we have assumed that the packing fraction is constant within the granular material, assumption that can be checked *a posteriori*. For that purpose, we can use Eq. 6 (with $\zeta = 0.2$) and the expressions of I derived in Sect. 4 for the shear plane flow and in Sect. 5 for the Poiseuille flow and see whether the variations of Φ are important or not. Figure 11 reports packing fraction profiles for the shear plane flow (a) and the Poiseuille flow (b). For the former geometry the

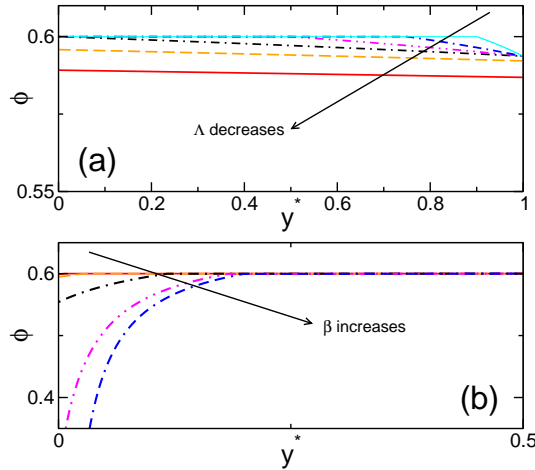


Fig. 11 The variations of the packing fraction profiles versus y^* is found to be almost constant for the shear plane flow (a). The values of Λ are $\Lambda = 2.5, 5, 10, 20, 40$ and 100 . On the contrary, for the Poiseuille flow, it displays important variations close to the sidewalls for important pressure gradients (b).

packing fraction varies slightly ($< 2\%$) justifying the approach used in this work. This is not surprising since, as mentioned above, our results agree with numerical and experimental results. In the Poiseuille geometry, at low β , the packing fraction variations are also weak. On the contrary, when β is increased, low values of the packing fraction are found close to sidewalls. Therefore, the approach used above is no more valid for those conditions (i.e. close to boundaries at high values of β). As mentioned above this was expected since the $\mu(I)$ rheology does not take into account the energy flux which are important close to the boundaries. Note however that the aforementioned conclusions obtained in that geometry at low β remain fully valid.

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References

1. R. Delannay, M. Louge, P. Richard, N. Taberlet, and A. Valance. Towards a theoretical picture of dense granular flows down inclines. *Nature Materials*, 6:99–108, 2007.
2. J. T. Jenkins and M. W. Richman. Kinetic theory for plane flows of a dense gas of identical, rough, inelastic, circular disks. *Physics of Fluids*, 28(12):3485–3494, 1985.
3. GDR MiDi. On dense granular flows. *The European Physical Journal E: Soft Matter and Biological Physics*, 14:341–365, 2004. 10.1140/epje/i2003-10153-0.
4. F. da Cruz, S. Emam, M. Prochnow, Jean-Noël Roux, and F. Chevoir. Rheophysics of dense granular materials: Discrete simulation of plane shear flows. *Phys. Rev. E*, 72:021309, Aug 2005.

5. I. Iordanoff and M. M. Khonsari. Granular lubrication: Toward an understanding of the transition between kinetic and quasi-fluid regime. *Journal of Tribology*, 126(1):137–145, 2004.
6. P. Jop, Y. Forterre, and O. Pouliquen. A constitutive law for dense granular flows. *Nature*, 441:727–730, June 2006.
7. N. Taberlet, P. Richard, J. T. Jenkins, and R. Delannay. Density inversion in rapid granular flows: the supported regime. *The European Physical Journal E: Soft Matter and Biological Physics*, 22(1):17–24, 2007.
8. Bereket Yohannes and K. M. Hill. Rheology of dense granular mixtures: Particle-size distributions, boundary conditions, and collisional time scales. *Phys. Rev. E*, 82:061301, Dec 2010.
9. P.-Y. Lagr  e, L. Staron, and S. Popinet. The granular column collapse as a continuum: validity of a two-dimensional navier-stokes model with a $\mu(i)$ -rheology. *Journal of Fluid Mechanics*, 686:378–408, 2011.
10. Stuart B. Savage. The mechanics of rapid granular flows. volume 24 of *Advances in Applied Mechanics*, pages 289 – 366. Elsevier, 1984.
11. Christophe Ancey, Philippe Coussot, and Pierre Evesque. A theoretical framework for granular suspensions in a steady simple shear flow. *Journal of Rheology*, 43(6):1673–1699, 1999.
12. P. Jop, Y. Forterre, and O. Pouliquen. Crucial role of sidewalls in granular surface flows: consequences for the rheology. *Journal of Fluid Mechanics*, 541:167–192, 2005.
13. P.-P. Cortet, D. Bonamy, F. Daviaud, O. Dauchot, B. Dubrulle, and M. Renouf. Relevance of visco-plastic theory in a multi-directional inhomogeneous granular flow. *EPL (Europhysics Letters)*, 88(1):14001, 2009.
14. Nicolas Brodu, Patrick Richard, and Renaud Delannay. Shallow granular flows down flat frictional channels: Steady flows and longitudinal vortices. *Phys. Rev. E*, 87:022202, Feb 2013.
15. Y. Forterre and O. Pouliquen. Flows of dense granular media. *Annu. Rev. Fluid Mech.*, 40:1–24, 2008.
16. J. Jenkins and D. Berzi. Dense inclined flows of inelastic spheres: tests of an extension of kinetic theory. *Granular Matter*, 12:151–158, 2010. 10.1007/s10035-010-0169-8.
17. I. S. Aranson and L. S. Tsimring. Continuum description of avalanches in granular media. *Phys. Rev. E*, 64:020301, Jul 2001.
18. C. Josserand, P.-Y. Lagr  e, and D. Lhuillier. Stationary shear flows of dense granular materials: a tentative continuum modelling. *The European Physical Journal E: Soft Matter and Biological Physics*, 14:127–135, 2004. 10.1140/epje/i2003-10141-4.
19. P. Mills, D. Loggia, and M. Tixier. Model for a stationary dense granular flow along an inclined wall. *EPL (Europhysics Letters)*, 45(6):733, 1999.
20. M. Y. Louge. Model for dense granular flows down bumpy inclines. *Phys. Rev. E*, 67:061303, Jun 2003.
21. D. Berzi, C. G. di Prisco, and D. Vescovi. Constitutive relations for steady, dense granular flows. *Phys. Rev. E*, 84:031301, Sep 2011.
22. Philippe Rib  re, Patrick Richard, Renaud Delannay, Daniel Bideau, Masahiro Toiya, and Wolfgang Losert. Effect of rare events on out-of-equilibrium relaxation. *Phys. Rev. Lett.*, 95(26):268001, Dec 2005.
23. K. Nichol, A. Zanin, R. Bastien, E. Wandersman, and M. van Hecke. Flow-induced agitations create a granular fluid. *Phys. Rev. Lett.*, 104:078302, Feb 2010.
24. K. A. Reddy, Y. Forterre, and O. Pouliquen. Evidence of mechanically activated processes in slow granular flows. *Phys. Rev. Lett.*, 106:108301, Mar 2011.
25. Olivier Pouliquen and Yoel Forterre. A non-local rheology for dense granular flows. *Phil. Trans. R. Soc. A*, 367:5091–5107, 2009.
26. Ken Kamrin and Georg Koval. Nonlocal constitutive relation for steady granular flow. *Phys. Rev. Lett.*, 108:178301, Apr 2012.
27. T. B  rzs  nyi, R. E. Ecke, and J. N. McElwaine. Patterns in flowing sand: Understanding the physics of granular flow. *Phys. Rev. Lett.*, 103(17):178302, Oct 2009.
28. Alex J. Holyoake and Jim N. McElwaine. High-speed granular chute flows. *Journal of Fluid Mechanics*, 710:35–71, 2012.
29. F. da Cruz. *  coulement des grains sec: Frottement et blocage*. PhD thesis,   cole Nationale des Ponts et chauss  es, 2004.
30. L. E. Silbert, D. Ertas, G. S. Grest, T. C. Halsey, D. Levine, and S. J. Plimpton. Granular flow down an inclined plane: Bagnold scaling and rheology. *Phys. Rev. E*, 64(5):051302, Oct 2001.

31. E. Buckingham. On physically similar systems; illustrations of the use of dimensional equations. *Phys. Rev.*, 4:345–376, Oct 1914.
32. Y. Amarouchene, J. F. Boudet, and H. Kellay. Dynamic sand dunes. *Phys. Rev. Lett.*, 86:4286–4289, May 2001.
33. T. S. Komatsu, S. Inagaki, N. Nakagawa, and S. Nasuno. Creep motion in a granular pile exhibiting steady surface flow. *Phys. Rev. Lett.*, 86:1757–1760, Feb 2001.
34. J.-P. Bouchaud, M. E. Cates, J. Ravi Prakash, and S. F. Edwards. Hysteresis and metastability in a continuum sandpile model. *Phys. Rev. Lett.*, 74:1982–1985, Mar 1995.
35. T. Boutreux, E. Raphaël, and P.-G. de Gennes. Surface flows of granular materials: a modified picture for thick avalanches. *Phys. Rev. E*, 58:4692–4700, Oct 1998.
36. N. Taberlet, P. Richard, E. Henry, and R. Delannay. The growth of a super stable heap: An experimental and numerical study. *EPL (Europhysics Letters)*, 68(4):515–521, 2004.
37. A. Mangeney, L.S. Tsimring, D. Volfson, I.S. Aranson, and F. Bochut. Avalanche mobility induced by the presence of an erodible bed and associated entrainment. *Geophysical Research Letters*, 34:L22401, 2007.
38. Nicolas Taberlet, Patrick Richard, and Renaud Delannay. The effect of sidewall friction on dense granular flows. *Computers & Mathematics with Applications*, 55(2):230 – 234, 2008. Modeling Granularity, Modeling Granularity.
39. J. Crassous, J.-F. Metayer, P. Richard, and C. Laroche. Experimental study of a creeping granular flow at very low velocity. *Journal of Statistical Mechanics: Theory and Experiment*, 2008(03):P03009, 2008.
40. P. Richard, A. Valance, J.-F. Métayer, P. Sanchez, J. Crassous, M. Louge, and R. Delannay. Rheology of confined granular flows: Scale invariance, glass transition, and friction weakening. *Phys. Rev. Lett.*, 101:248002, Dec 2008.
41. R. Blumenfeld, S. Edwards, and M. Schwartz. da vinci fluids, catch-up dynamics and dense granular flow. *The European Physical Journal E: Soft Matter and Biological Physics*, 32:333–338, 2010. 10.1140/epje/i2010-10628-9.